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Cycles and Paths Embedded in Varietal Hypercubes *

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Abstract

The varietal hypercube VQ_n is a variant of the hypercube Q_n and has better properties than Q_n with the same number of edges and vertices. This paper shows that every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5, and every pair of vertices with distance d is connected by paths of every length from d to $2^n - 1$ except 2 and 4 if $d = 1$.

Keywords Combinatorics, cycle, path, varietal hypercube, pancyclicity, panconnectivity

AMS Subject Classification: 05C38 90B10

1 Introduction

The hypercube network Q_n has proved to be one of the most popular interconnection networks since it has a simple structure and has many nice properties. As a variant of Q_n, the varietal hypercube VQ_n, proposed by Cheng and Chuang [1] in 1994, has many properties similar or superior to Q_n. For example, the connectivity and restricted connectivity of VQ_n and Q_n are the same (see Wang and Xu [4]), while, all the diameter and the average distance, fault-diameter and wide-diameter of VQ_n are smaller than that of the hypercube (see Cheng and Chuang [1], Jiang *et al.* [3]).

Several topological structures of multicomputer systems are commonly used in various applications such as image processing and scientific computing. Among them, the most common structures are paths and cycles. Embedding these structures in various well-known networks, such as Q_n, have been extensively investigated in the literature (see, for example, a survey by Xu and ma [5]). However, embedding these structures in VQ_n has been not investigated as yet. In this paper, we show

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that VQ_n should be capable of embedding these structures. Main results can be stated as follows.

Every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5, and every pair of vertices with distance d is connected by paths of every length from d to $2^n - 1$ except 2 and 4 if $d = 1$.

The proofs of these results are in Section 3. The definition and some basic properties of VQ_n are given in Section 2.

2 Definitions and Lemmas

We follow [7] for graph-theoretical terminology and notation not defined here. A graph $G = (V, E)$ always means a simple and connected graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of G. For $uv \in E(G)$, we call u (resp. v) is a neighbor of v (resp. u). A *uv-path* is a sequence of adjacent vertices, written as $(v_0, v_1, v_2, \cdots, v_m)$, in which $u = v_0$, $v = v_m$ and all the vertices $v_0, v_1, v_2, \cdots, v_m$ are different from each other, u and v is called the *end-vertices* of P. If $u = v$, then a uv-path P is called a *cycle*. The *length* of a path P, denoted by $\varepsilon(P)$, is the number of edges in P. The length of a shortest uv-path in G is called the *distance* between u and v in G, denoted by $d_G(u, v)$. For a path $P = (v_0, v_1, \cdots, v_i, v_{i+1}, \cdots, v_m)$, we can write $P = P(v_0, v_i) + v_i v_{i+1} + P(v_{i+1}, v_m)$, and the notation $P - v_i v_{i+1}$ denotes the subgraph obtained from P by deleting the edge $v_i v_{i+1}$.

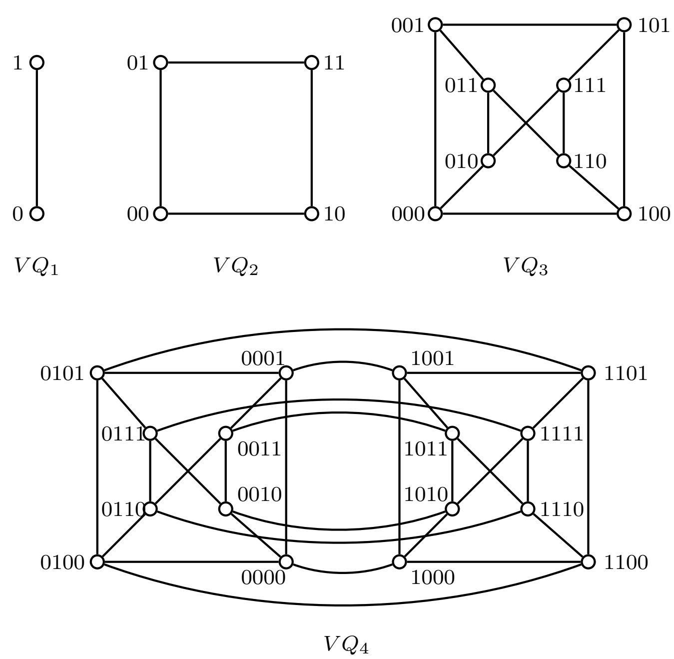

Figure 1: The varietal hypercubes VQ_1, VQ_2, VQ_3 and VQ_4

The *n-dimensional varietal hypercube* VQ_n is the labeled graph defined recursively as follows. VQ_1 is the complete graph of two vertices labeled with 0 and 1, respectively. Assume that VQ_{n-1} has been constructed. Let VQ^0_{n-1} (resp. VQ^1_{n-1}) be a labeled graph obtained from VQ_{n-1} by inserting a zero (resp. 1) in front of each vertex-labeling in VQ_{n-1}. For $n > 1$, VQ_n is obtained by joining vertices in

VQ^0_{n-1} and VQ^1_{n-1}, according to the rule: a vertex $x = 0x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1$ in VQ^0_{n-1} and a vertex $y = 1y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ in VQ^1_{n-1} are adjacent in VQ_n if and only if

1) $x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1 = y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ if $n \neq 3k$, or

2) $x_{n-3}\cdots x_2x_1 = y_{n-3}\cdots y_2y_1$ and $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in I$ if $n = 3k$, where $I = \{(00, 00), (01, 01), (10, 11), (11, 10)\}$.

Figure 1 shows the examples of varietal hypercubes VQ_n for $n = 1, 2, 3$ and 4.

The edges of Type 2) are referred to as *crossing edges* when $(x_{n-1}x_{n-2}, y_{n-1}y_{n-2}) \in \{(10, 11), (11, 10)\}$. All the other edges are referred to as *normal edges*.

The varietal hypercube VQ_n is proposed by Cheng and Chuang [1] as an attractive alternative to the n-dimensional hypercube Q_n when they are used to model the topological structure of a large-scale parallel processing system. Like Q_n, VQ_n is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges.

For convenience, we express VQ_n as $VQ_n = L \odot R$, where $L = VQ^0_{n-1}$ and $R = VQ^1_{n-1}$, and denote by x_Lx_R the n-transversal edge joining $x_L \in L$ and $x_R \in R$. The recursive structure of VQ_n gives the following simple properties.

Lemma 2.1 *Let $VQ_n = L \odot R$ with $n \geq 1$. Then VQ_n contains no triangles and every vertex $x_L \in L$ has exactly one neighbor x_R in R joined by the n-transversal edge x_Lx_R.*

Lemma 2.2 *Let $VQ_n = L \odot R$ and xy be an n-transversal edge in VQ_n with $x \in L$ and $y \in R$. For $n \geq 3$, let $x = 0abx_{n-3}\cdots x_1$ and $\beta = x_{n-3}\cdots x_1$. Then $y = 1a'b'\beta$, where $ab = a'b'$ if xy is a normal edge, and $(ab, a'b') = (1b, 1\bar{b})$ if xy is a crossing edge, where $\bar{b} = \{0, 1\} \setminus b$.*

Lemma 2.3 *Any edge in VQ_n $(n \geq 2)$ is contained in a cycle of length 4.*

Proof. Clearly, the conclusion is true for $n = 2$. Assume $n \geq 3$ and let xy be any edge in VQ_n. Then by definition of VQ_n there is some m with $2 \leq m \leq n$ such that xy is an m-transversal edge. Let $VQ_m = L \odot R$, $x \in L$ and $y \in R$.

If xy is a normal edge, let u_L be a neighbor of x in L and u_R be the neighbor of u_L in R, then y and u_R are adjacent and so (x, u_L, u_R, y) is a cycle of length 4.

If xy is a crossing edge, let $x = 01b\beta$, then $y = 11\bar{b}\beta$. Choose $u_L = 01\bar{b}\beta$. Then $u_R = 11b\beta$ by Lemma 2.2, and so (x, u_L, u_R, y) is a cycle of length 4. ∎

Lemma 2.4 *Any n-transversal edge must be contained in some cycle of length 5 unless $n \neq 3k$ for $k \geq 1$.*

Proof. Let $VQ_n = L \odot R$ and xy be an n-transversal edge in VQ_n, where $x \in L$ and $y \in R$. We first prove that xy is not contained in any cycle of length 5 if $n \neq 3k$ for $k \geq 1$. The conclusion is true for $n = 1$ or 2 clearly. Assume $n \geq 3$ below.

Suppose that there is a cycle $C = (x, u, z, v, y)$ of length 5 containing the edge xy. Then C contains two n-transversal edges. Since $n \neq 3k$, xy is a normal edge. Let $x = 0ab\beta$, where $\beta = x_{n-3}\cdots x_1$. Then $y = 1ab\beta$. Since every vertex in L has exactly one neighbor in R by Lemma 2.1, $u \in L$ and $v \in R$. Without loss of generality, assume $z \in L$. Then x and z differ in exactly two positions. Without loss of generality, let $z = 0\bar{a}\bar{b}\beta$. Since zv is an n-transversal edge and $n \neq 3k$, $v = 1\bar{a}\bar{b}\beta$.

Thus, y and v differ in exactly two positions, which implies that y and v are not adjacent, a contradiction.

We now show that the n-transversal edge xy must be contained in some cycle of length 5 if $n = 3k$ for $k \geq 1$ by constructing such a cycle. Let $x = 0ab\beta \in L$ and $y = 1a'b'\beta \in R$, where $(ab, a'b') \in I$. A required cycle $C = (x, u, z, v, y)$ can be constructed as follows.

If xy is a normal edge, then $ab = a'b' = 0b$. Let $u = 00\bar{b}\beta$, $z = 01\bar{b}\beta$ and $v = 11b\beta$ (where zv is a crossing edge).

If xy is a crossing edge, then $(ab, a'b') = (1b, 1\bar{b})$. Let $u = 01\bar{b}\beta$, $z = 00\bar{b}\beta$ and $v = 10\bar{b}\beta$ (where zv is a normal edge).

The lemma follows. ∎

Lemma 2.5 *Any n-transversal edge in VQ_n is contained in cycles of length* 6 *and* 7 *for $n \geq 3$.*

Proof. Let $VQ_n = L \odot R$ and xy be an n-transversal edge in VQ_n, where $x \in L$ and $y \in R$.

We first show that xy is contained in a cycle of length 6. By Lemma 2.3, there is a cycle C of length 4. Let $C = (x, u, v, y)$, where $u \in L$ and $v \in R$. Also by Lemma 2.3, there is a cycle C' of length 4 containing the xu in L. Clearly, $C \cap C' = \{xu\}$. Thus, $C \cup C' - xu$ is a cycle of length 6 containing the edge xy.

We now show that xy is contained in a cycle of length 7. If $n = 3k$ for $k \geq 1$ then, by Lemma 2.4, there is a cycle C of length 5 containing the edge xy. Let $C = (x, u, z, v, y)$, where $x, u, z \in L$ and $v \in R$, without loss of generality. By Lemma 2.3, there is a cycle C' of length 4 containing the edge yv in R. Clearly, $C \cap C' = \{yv\}$. Thus $C \cup C' - yv$ is a cycle of length 7 containing the edge xy.

Assume $n \neq 3k$ for $k \geq 1$ below. In this case, all n-transversal edges are normal edges. We can choose a cycle $C = (x, u, v, y)$ such that the edge xu lies on some subgraph H that is isomorphic to VQ_3. By Lemma 2.4, there is a cycle C' of length 5 containing the edge xu in $H \subseteq L$. Then $C \cup C' - xu$ is a cycle of length 7 containing the edge xy.

The lemma follows. ∎

The *n-dimensional crossed cube* CQ_n is such a graph, its vertex-set is the same as VQ_n, two vertices $x = x_n \cdots x_2 x_1$ and $y = y_n \cdots y_2 y_1$ are linked by an edge if and only if there exists some j ($1 \leq j \leq n$) such that

(a) $x_n \cdots x_{j+1} = y_n \cdots y_{j+1}$,
(b) $x_j \neq y_j$,
(c) $x_{j-1} = y_{j-1}$ if j is even, and
(d) $(x_{2i}x_{2i-1}, y_{2i}y_{2i-1}) \in I$ for each $i = 1, 2, \cdots, \lceil \frac{1}{2} j \rceil - 1$.

By definitions, $VQ_n \cong CQ_n$ for each $n = 1, 2, 3$. The following results on CQ_n are used in the proofs of our main results for $n = 3$.

Lemma 2.6 (Fan *et al.* [2], Xu and Ma [6], Yang and Megson [8]) *For any two vertices x and y with distance d in CQ_n with $n \geq 2$, CQ_n contains xy-paths of every length from d to $2^n - 1$ except 2 when $d = 1$.*

Lemma 2.7 *For $n \geq 3$ and any integer ℓ with $2^n - 2 \leq \ell \leq 2^n - 1$, there exists an xy-path of length ℓ between any pair of vertices x and y in VQ_n.*

Proof. We proceed by induction on $n \geq 3$. By Lemma 2.6, the conclusion is true for $n = 3$ since $VQ_3 \cong CQ_3$. Assume the induction hypothesis for $n-1$ with $n \geq 4$. Let $VQ_n = L \odot R$, x and y be two distinct vertices in VQ_n.

If $x, y \in L$ (or R) then, by the induction hypothesis, there exists an xy-path P_L of length ℓ_0 in L, where $\ell_0 \in \{2^{n-1}-2, 2^{n-1}-1\}$. Let u be the neighbor of y in P_L, u_R and y_R be the neighbors of u and y in R, respectively. By the induction hypothesis, there exists a $u_R y_R$-path P_R of length $2^{n-1}-1$ in R. Then $P_L - uy + uu_R + P_R + y_R y$ is an xy-path of length $\ell_0 + 2^{n-1}$ in VQ_n.

If $x \in L$ and $y \in R$, let u be a vertex in L rather than x such that its neighbor u_R in R is different from y, then, by the induction hypothesis, there exist an xu-path P_L of length ℓ'_0 in L and a $u_R y$-path P_R of length $2^{n-1}-1$ in R, where $\ell'_0 \in \{2^{n-1}-2, 2^{n-1}-1\}$. Then $P_L + uu_R + P_R$ is an xy-path of length $\ell'_0 + 2^{n-1}$ in VQ_n.

The lemma follows. ∎

Lemma 2.8 *Let $VQ_n = L \odot R$, x_L and y_L be two vertices in L. Then $d_L(x_L, y_L) = d_R(x_R, y_R)$ if $n \neq 3k$ and $|d_L(x_L, y_L) - d_R(x_R, y_R)| \leq 2$ if $n = 3k$ for $k \geq 1$.*

Proof. Without loss of generality, assume $d_L(x_L, y_L) \leq d_R(x_R, y_R)$. Let P_L be a shortest $x_L y_L$-path in L and P_R a path in R obtained from P_L by replacing the first position 0 by 1 in every vertices. Clearly, $\varepsilon(P_R) = \varepsilon(P_L)$.

Note that for an edge $u_L v_L$ in P_L, if $u_L v_R$ is a crossing edge, then $v_L u_R$ is also a crossing edge. For convenience, we call the edge $u_L v_L$ an *induced crossing edge*, u_L and v_L *induced crossing vertices*.

If both x and y are not induced crossing vertices, then P_R is an $x_R y_R$-path in R, and so $d_R(x_R, y_R) \leq \varepsilon(P_R) = d_L(x_L, y_L)$, and so $d_R(x_R, y_R) = d_L(x_L, y_L)$. Assume below that $\{x, y\}$ contains induced crossing vertices. Then $n = 3k$.

Let x be an induced crossing vertex, xu_L an induced crossing edge. Then, x_R is not an end-vertex of P_R, while u_R is an end-vertex of P_R. Similarly, if y is an induced crossing vertex, yv_L an induced crossing edge, then y_R is not an end-vertex of P_R, while v_R is an end-vertex of P_R. Thus, an $x_R y_R$-path $P'_R \subseteq P_R$ has length

$$\varepsilon(P'_R) = \varepsilon(P_L) - \begin{cases} 0 & \text{if neither } x \text{ and } y \text{ are induced crossing vertices;} \\ 1 & \text{if either } x \text{ or } y \text{ is an induced crossing vertex;} \\ 2 & \text{if both } x \text{ and } y \text{ are induced crossing vertices,} \end{cases}$$

and $d_R(x_R, y_R) \leq \varepsilon(P'_R)$. If $d_R(x_R, y_R) \leq d_L(x_L, y_L) - 3$ then, using the above method, we can prove that there is an $x_L y_L$-path P'_L with length $\varepsilon(P'_L) \leq d_R(x_R, y_R) + 2$, from which we have $d_L(x_L, y_L) \leq \varepsilon(P'_L) \leq d_R(x_R, y_R) + 2 \leq d_L(x_L, y_L) - 1$, a contradiction. Thus, $d_R(x_R, y_R) \geq d_L(x_L, y_L) - 2$. And so $|d_L(x_L, y_L) - d_R(x_R, y_R)| \leq 2$. The lemma follows. ∎

Corollary 2.9 *Let $VQ_n = L \odot R$, x and y be two vertices in H, where $H \in \{L, R\}$. Then $d_H(x, y) = d_{VQ_n}(x, y)$.*

Proof. Let x and y be in L and P a shortest xy-path in VQ_n. If $P \cap R \neq \emptyset$, then $P \cap L$ consists of several sections of P. Without loss of generality, assume that $P \cap L$ consists of two sections, P_{xu_L} and $P_{v_L y}$. Then $u_R v_R$-section $P_{u_R v_R}$ of P from u_R to

v_R is in R. By Lemma 2.8, $d_L(u_L, v_L) \leq d_R(u_R, v_R) + 2 = \varepsilon(P_{u_R v_R}) + 2$. Since P is a shortest xy-path in VQ_n, we have that

$$\begin{aligned} d_{VQ_n}(x,y) &\leq d_L(x,y) = \varepsilon(P_{xu_L}) + d_L(u_L, v_L) + \varepsilon(P_{v_L y}) \\ &\leq \varepsilon(P_{xu_L}) + \varepsilon(P_{u_R v_R}) + 2 + \varepsilon(P_{v_L y}) \\ &= \varepsilon(P) = d_{VQ_n}(x,y), \end{aligned}$$

which implies $d_L(x,y) = d_{VQ_n}(x,y)$. The corollary follows. ∎

Corollary 2.10 *Let $VQ_n = L \odot R$, $x \in L$ and $y \in R$. Then there is an n-transversal edge $u_L u_R$ such that $d_{VQ_n}(x,y) = d_L(x,u_L) + 1 + d_R(u_R, y)$.*

3 Main Results

A graph G of order n is said to be *ℓ-pancyclic* (resp. *ℓ-vertex-pancyclic, ℓ-edge-pancyclic*) if it contains (resp. each of its vertices, edges is contained in) cycles of every length from ℓ to n. Clearly, an ℓ-edge-pancyclic graph must is ℓ-vertex-pancyclic and ℓ-pancyclic.

We consider edge-pancyclity of VQ_n. Since VQ_n contains no triangles, any edge is not contained in a cycle of length 3. Lemma 2.3 shows that any edge in VQ_n $(n \geq 2)$ is contained in a cycle of length 4. Lemma 2.4 shows that any n-transversal edge is not contained in a cycle of length 5 if $n \neq 3k$ for $k \geq 1$. In general, we have the following result.

Theorem 3.1 *For $n \geq 2$, every edge of VQ_n is contained in cycles of every length from 4 to 2^n except 5 and, hence, VQ_n is 6-edge-pancyclic for $n \geq 3$.*

Proof. By Lemma 2.3, we only need to show that every edge of VQ_n is contained in cycles of every length from 6 to 2^n for $n \geq 3$. Let ℓ be an integer with $6 \leq \ell \leq 2^n$ and xy be an edge in VQ_n. In order to prove the theorem, we only need to show that xy lies on a cycle of length ℓ. We proceed by induction on $n \geq 3$.

Since $VQ_3 \cong CQ_3$, by Lemma 2.6, the conclusion is true for $n = 3$. Assume the induction hypothesis for $n-1$ with $n \geq 4$. Let $VQ_n = L \odot R$. There are two cases.

Case 1 $x, y \in L$ or $x, y \in R$. Without loss of generality, let $x, y \in L$.

By the induction hypothesis, we only need to consider ℓ with $2^{n-1} + 1 \leq \ell \leq 2^n$.

If $\ell = 2^{n-1} + 1$, then let x_R and y_R be the neighbors of x and y in R, respectively. By Lemma 2.7, there exists an $x_R y_R$-path $P_{x_R y_R}$ of length $2^{n-1} - 2$ in R. Then $xx_R + P_{x_R y_R} + y_R y + xy$ is a cycle of length $2^{n-1} + 1$.

If $2^{n-1} + 2 \leq \ell \leq 2^n$, let $\ell_0 = \ell - 2^{n-1} - 1$, then $1 \leq \ell_0 \leq 2^{n-1} - 1$. By Lemma 2.7, there exists a cycle C of length 2^{n-1} containing the edge xy in L. We choose an xz-path P_{xz} of length ℓ_0 in C that contains xy. Let x_R and z_R be the neighbors of x and z in R, respectively. By Lemma 2.7, there exists an $x_R z_R$-path $P_{x_R z_R}$ of length $2^{n-1} - 1$ in R. Then $xx_R + P_{x_R z_R} + z_R z + P_{xz}$ is a cycle of length ℓ containing the edge xy in VQ_n.

Case 2 $x \in L$ and $y \in R$.

In this case, xy is an n-transversal edge. By Lemma 2.5, the conclusion is true for each $\ell = 6, 7$. Assume $\ell \geq 8$ below.

If $\ell \le 2^{n-1}+2$, let $\ell_0 = \ell - 2$, then $6 \le \ell_0 \le 2^{n-1}$. By Lemma 2.3, there exists a 4-cycle $C = (x, u_L, u_R, y)$. By the induction hypothesis, there exists a cycle C_L of length ℓ_0 that contains xu_L in L. Then, $C \cap C_L = \{xu_L\}$, and so $C \cap C_L - \{xu_L\}$ is a cycle of length ℓ containing xy.

If $2^{n-1}+3 \le \ell \le 2^n$, let $\ell_0 = \ell - 2^{n-1} - 1$, then $2 \le \ell_0 \le 2^{n-1} - 1$. Choose a vertex u in L rather than x, By Lemma 2.7, there exists an xu-path P_{xu} of length $2^{n-1} - 1$ in L, from which we can choose an xz-path P_{xz} of length ℓ_0. Let z_R be the neighbor of z in R. By Lemma 2.7, there exists a z_Ry-path P_{z_Ry} of length $2^{n-1} - 1$. Thus, $P_{xz} + zz_R + P_{z_Ry} + xy$ is a cycle of length ℓ containing xy in VQ_n.

The theorem follows. ∎

A graph G of order n is said to be *panconnected* if for any two distinct vertices x and y with distance d in G there are xy-paths of every length from d to $n-1$.

We consider panconnectivity of VQ_n. Since VQ_n contains no triangles, there exist no xy-paths of length two if x and y are adjacent. Lemma 2.4 shows that there exist no xy-paths of length 4 if xy is an n-transversal edge in VQ_n if $n \ne 3k$ for $k \ge 1$. In general, we have the following result.

Theorem 3.2 *For $n \ge 3$, any two vertices x and y in VQ_n with distance d, there exist xy-paths of every length from d to $2^n - 1$ except $2, 4$ if $d = 1$.*

Proof. Let x and y be any two vertices in VQ_n with distance d. First, we note that if $d = 1$ then the theorem is true by Theorem 3.1. In the following discussion, we always assume $d \ge 2$. We only need to prove that there exist xy-paths of every length from $d+1$ to $2^n - 1$.

We proceed by induction on $n \ge 3$. Since $VQ_3 \cong CQ_3$, by Lemma 2.6, the conclusion is true for $n = 3$. Assume the induction hypothesis for $n-1$ with $n \ge 4$. Let $VQ_n = L \odot R$.

Case 1. $x, y \in L$ or $x, y \in R$. Without loss of generality, let $x, y \in L$.

By Corollary 2.9, $d_L(x, y) = d$. By the induction hypothesis, we only need to consider ℓ with $2^{n-1} \le \ell \le 2^n - 1$.

If $2^{n-1} \le \ell \le 2^{n-1} + 1$, then $2^{n-1} - 2 \le \ell - 2 \le 2^{n-1} - 1$. Let x_R and y_R be the neighbors of x and y in R, respectively. By Lemma 2.7, there exists an x_Ry_R-path P_R of length $\ell - 2$ in R. Then $x_Ry_R + P_R + yy_R$ is an xy-path of length ℓ in VQ_n.

If $2^{n-1} + 2 \le \ell \le 2^n - 1$, let $\ell_0 = \ell - 2^{n-1} - 1$. then $1 \le \ell_0 \le 2^{n-1} - 2$. By Lemma 2.7, there exists an xy-path P_{xy} of length $2^{n-1} - 1$ in L. We choose an xz-path P_{xz} of length ℓ_0 in P_{xy}. Clearly, $z \notin \{x, y\}$. Let z_R and y_R be the neighbors of z and y in R, respectively. By Lemma 2.7, there exists a z_Ry_R-path P_R of length $2^{n-1} - 1$ in R. Then $P_{xz} + zz_R + P_R + yy_R$ is an xy-path of length ℓ in VQ_n.

Case 2. $x \in L$ and $y \in R$.

By Corollary 2.10, there is a shortest xy-path P_{xy} in VQ_n such that $P_{xy} = P_{xu_L} + u_Lu_R + Pu_Ry$, where $u_L \in L$ and $u_R \in R$, $\varepsilon(P_{xu_L}) = d_L(x, u_L)$ and $\varepsilon(P_{u_Ry}) = d_R(u_R, y)$. Thus, $d = \varepsilon(P_{xu_L}) + 1 + \varepsilon(P_{u_Ry}) = d_L(x, u_L) + 1 + d_R(u_R, y)$. Since $d \ge 2$, without loss of generality, assume $d_L(x, u_L) \ge d_R(u_R, y)$.

If $d+1 \le \ell \le 2^{n-1}$, let $\ell_0 = \ell - d_R(u_R, y) - 1$, then $d_L(x, u_L) + 1 \le \ell_0 \le 2^{n-1} - 1$. By the induction hypothesis, there exists an xu_L-path P' of length ℓ_0 in L. Then $P' + u_Lu_R + P_{u_Ry}$ is an xy-path of length ℓ in VQ_n.

If $2^{n-1}+1 \leq \ell \leq 2^n-1$, let $\ell_0 = \ell - 2^{n-1}$, then $1 \leq \ell_0 \leq 2^{n-1}-1$. Let y_L be the neighbor of y in L. Then $y_L \neq x$ since x and y are not adjacent. By Lemma 2.7, there exists an xy_L-path P_{xy_L} of length $2^{n-1}-1$ in L. We choose an xz-path P_{xz} of length ℓ_0 in P_{xy_L}. Let z_R be the neighbor of z in R. By Lemma 2.7, there exists a z_Ry-path P_{z_Ry} of length $2^{n-1}-1$ in R. Then $P_{xz} + zz_R + P_{z_Ry}$ is an xy-path of length ℓ in VQ_n.

The theorem follows. ∎

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